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POSITIVE AND ASYMPTOTICALLY STABLE TIME-VARYING
CONTINUOUS-TIME LINEAR SYSTEMS AND ELECTRICAL CIRCUITS

The positivity and asymptotic stability of time-varying continuous-time linear systems and electrical circuits are addressed. Necessary and sufficient conditions for the positivity and asymptotic stability of the systems and electrical circuits are established. It is shown that there exists a large class of positive and asymptotically stable electrical circuits with time-varying parameters. Examples of positive and asymptotically stable electrical circuits are presented.

1. INTRODUCTION

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [3, 4]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The Lyapunov, Bohl and Perron exponents and stability of time-varying discrete-time linear systems have been investigated in [1, 2, 5-9]. The positivity and stability of fractional time varying discrete-time linear systems have been addressed in [15, 16, 22, 24] and the stability of continuous-time linear systems with delays in [21]. The fractional positive linear systems have been analyzed in [9-16]. The positive electrical circuits and their reachability have been considered in [11, 14] and the controllability and observability in [19, 20]. The stability and stabilization of positive fractional linear systems by state-feedbacks have been analyzed in [12]. Positive linear systems consisting of n subsystems with different fractional orders have been analyzed in [18]. The normal positive electrical circuits has been introduced in [20].

In this paper positivity and asymptotic stability of time-varying continuous-time linear systems and electrical systems will be addressed.

The paper is organized as follows. In section 2 the solution to the time-varying linear systems and their properties are recalled. Necessary and sufficient conditions for the positivity of time-varying continuous-time linear systems are established in section 3 and of asymptotic stability in section 4. The positive electrical circuits with time-varying parameter are addressed in section 5. Concluding remarks are given in section 6.

The following notation will be used: $\mathbb{R}$ - the set of real numbers, $\mathbb{R}^{m\times n}$ - the set of $n \times m$ real matrices, $\mathbb{R}_+^{m\times n}$ - the set of $n \times m$ matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$, $M_n$ - the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), $I_n$ - the $n \times n$ identity matrix, $T$ - denotes the transposition of matrix (vector).

2. PRELIMINARIES

Consider the matrix linear differential equation with time-varying coefficients [23, 25]

$$\frac{dX}{dt} = A(t)X$$

where $X = X(t) \in \mathbb{R}^{n\times n}$ and $A(t) \in \mathbb{R}^{n\times n}$ with entries $a_{ij}(t)$ being continuous-time functions of time $t \in [0, +\infty)$.

To solve the equation (1) the Picard method will be used

$$\frac{dX_k}{dt} = A(t)X_{k-1} \text{ for } k = 1, 2, \ldots.$$  

From (2) we obtain

$$X_k = I_n + \int_{t_0}^{t} A(\tau)X_{k-1}(\tau) d\tau$$

where $X(t_0) = I_n$.

Using (3) for $k = 1, 2, \ldots$ and

$$\Omega_{t_0}(A) = I_n + \int_{t_0}^{t} A(\tau)d\tau + \int_{t_0}^{t} A(\tau_1)d\tau_1 + \int_{t_0}^{t} A(\tau_2)d\tau_2 + \ldots$$

we can write the solution of (1) in the form

$$X(t) = \Omega_{t_0}(A)X_0$$

and $X_0 = X(t_0)$ is the initial condition.

It is easy to show [25] that

$$\Omega_{t_0}(A) = \Omega_{t_1}(A)\Omega_{t_2}(A) \text{ for } t_0 < t_1 < t \in [0, +\infty).$$

Lemma 1. If the matrix $A(t) \in \mathbb{R}^{n\times n}$ satisfy the condition

$$A(t_1)A(t_2) = A(t_2)A(t_1) \text{ for } t_1, t_2 \in [t_0, t] \in [0, +\infty)$$

then

$$\Omega_{t_0}(A) = e^{\int_{t_0}^{t} A(\tau)d\tau}.$$  

Proof. is given in [23, 25].

Lemma 2. If $A_1 = A_1(t) \in \mathbb{R}^{n\times n}$ and $A_2 = A_2(t) \in \mathbb{R}^{n\times n}$, $t \in [0, +\infty)$, then

$$\Omega_{t_0}(A_1 + A_2) = \Omega_{t_0}(A_1)\Omega_{t_0}(A_2)$$

where

$$A = A(t) = [\Omega_{t_0}(A_1)]^{-1}A_2\Omega_{t_0}(A_1).$$

Proof. Let

$$X = X(t) = \Omega_{t_0}(A_1), \ Y = Y(t) = \Omega_{t_0}(A_1 + A_2).$$
Differentiating \( Y = XZ \) we obtain \( \frac{dY}{dt} = \frac{dX}{dt} Z + X \frac{dZ}{dt} \) and taking into account that 
\[
\frac{dY}{dt} = (A_1 + A_2)Y \quad \text{and} \quad \frac{dX}{dt} = A_1X
\]
we have \((A_1 + A_2)XZ = A_1XZ + X A_2 Z\) or
\[
A_2XZ = X \frac{dZ}{dt}.
\] (12)
Solving (12) we obtain
\[
\frac{dZ}{dt} = X^{-1} A_2 XZ
\] (13)
and
\[
Z = \Omega^t_0(X^{-1} A_2 X).
\] (14)
This completes the proof. □

**Lemma 3.** Let \( A \in \mathbb{R}^{n \times n} \) be a matrix with constant entries independent of time \( t \). If \( A(t) = A \) then
\[
\Omega^t_0(A) = e^{A(t-t_0)}.
\] (15)
Proof is given in [25].

Now let us consider the time-varying system described by the equation
\[
\dot{x}(t) = A(t)x + B(t)u,
\] (16)
where \( x = x(t) \in \mathbb{R}^n \), \( u = u(t) \in \mathbb{R}^m \) are the state and input vectors and \( A(t) \in \mathbb{R}^{n \times n} \), \( B(t) \in \mathbb{R}^{n \times m} \) are matrices with entries depending continuously on time \( t \in [0, +\infty) \).

**Lemma 4.** The solution of the equation (16) with initial condition \( x_0 = x(t_0) \in \mathbb{R}^n \) and input \( u(t) \in \mathbb{R}^m \) has the form
\[
x = \Omega^t_0(A)x(t_0) + \int_{t_0}^{t} K(t, \tau)B(\tau)u(\tau)d\tau
\] (17a)
where
\[
K(t, \tau) = \Omega^t_0(A)[\Omega^\tau_0(A)]^{-1}.
\] (17b)
Proof is given in [24].

### 3. POSITIVE TIME-VARYING CONTINUOUS-TIME LINEAR SYSTEMS

Consider the time-varying linear system
\[
\dot{x}(t) = A(t)x(t) + B(t)u(t)
\] (18a)
\[
y(t) = C(t)x(t) + D(t)u(t)
\] (18b)
where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), \( y(t) \in \mathbb{R}^p \) are the state, input and output vectors and \( A(t) \in \mathbb{R}^{n \times n} \), \( B(t) \in \mathbb{R}^{n \times m} \), \( C(t) \in \mathbb{R}^{p \times n} \), \( D(t) \in \mathbb{R}^{p \times m} \) are real matrices with entries depending continuously on time and \( \text{det}(A(t)) \neq 0 \) for \( t \in [0, +\infty) \).

**Definition 1.** The system (18) is called positive if \( x(t) \in \mathbb{R}^n_+ \), \( y(t) \in \mathbb{R}^p_+ \), \( t \in [0, +\infty) \) for any initial conditions \( x_0 \in \mathbb{R}^n_+ \) and all inputs \( u(t) \in \mathbb{R}^m_+ \), \( t \in [0, +\infty) \).

**Theorem 1.** Let \( A(t) \in \mathbb{R}^{n \times n}_+ \), \( t \in [0, +\infty) \). Then
\[
\Omega^t_0(A) = I_n + \int_{t_0}^{t} A(t)dt + \frac{1}{2} \int_{t_0}^{t} A(t)A(t)^Tdt + \cdots \in \mathbb{R}^{n \times n}_+
\] (19)
for \( t \geq t_0 \) and only if \( A(t) \in M_n \), \( t \in [0, +\infty) \).

**Proof.** Necessity. From (19) it follows that \( \Omega^t_0(A) \in \mathbb{R}^{n \times n}_+ \) for small value of \( t > t_0 \) only if \( A(t) \in M_n \).

Sufficiency. Let choose constant \( \lambda > 0 \) such that
\[
\lambda \geq \max_{1 \leq i \leq n} |a_{ii}(t)|,
\] (20)
where \( a_{ii}(t) \) is the \( i \)-th \( 1 \leq i \leq n \) diagonal entry of \( A(t) \). In this case if \( A(t) \in M_n \) then \( A(t) + I_n\lambda \in \mathbb{R}^{n \times n}_+ \) for \( t \in [0, +\infty) \). Taking into account that \( \Omega^t_0(A) = [A(t) + I_n\lambda]^{-1} - I_n\lambda \) and Lemmas 2, 3 for \( A(t) = -I_n\lambda \) and \( A_2(t) = A(t) + I_n\lambda \) we obtain
\[
\Omega^t_0(A) = \Omega^t_0(-I_n\lambda)\Omega^t_0(A_2)(\lambda) = e^{-\lambda t} \Omega^t_0(A_2)(\lambda) \quad (21)
\]
for \( t \in [0, +\infty) \), since
\[
\Omega^t_0(-I_n\lambda) = e^{-\lambda t} \Omega^t_0(-I_n\lambda)\quad (22)
\]
for \( t \in [0, +\infty) \) if \( A(t) \in M_n \). This completes the proof. □

**Theorem 2.** The time-varying linear system (18) is positive if and only if
\[
A(t) \in M_n \quad B(t) \in \mathbb{R}^{n \times m}_+ \quad C(t) \in \mathbb{R}^{p \times n}_+ \quad D(t) \in \mathbb{R}^{p \times m}_+ \quad t \in [0, +\infty).
\] (23)

**Proof.** Sufficiency. By Lemma 4 the solution of the equation (18a) is given by (17a) and \( x(t) \in \mathbb{R}^n_+ \), \( t \in [0, +\infty) \) if \( A(t) \in M_n \) and \( B(t) \in \mathbb{R}^{n \times m}_+ \) for \( t \in [0, +\infty) \) since form Theorem 2 we have \( \Omega^t_0(A) \in \mathbb{R}^{n \times n}_+ \), \( K(t, \tau) \in \mathbb{R}^{n \times n}_+ \) and by assumption \( x(t_0) \in \mathbb{R}^n_+ \), \( u(t) \in \mathbb{R}^m_+ \), \( t \in [0, +\infty) \).

From the equation (18b) we have \( y(t) \in \mathbb{R}^p_+ \), \( t \in [0, +\infty) \) since \( C(t) \in \mathbb{R}^{p \times n}_+ \), \( D(t) \in \mathbb{R}^{p \times m}_+ \) and \( x(t) \in \mathbb{R}^n_+, \quad u(t) \in \mathbb{R}^m_+, \quad t \in [0, +\infty) \).

Necessity. Let \( u(t) = 0 \) for \( t \in [0, +\infty) \) and \( x(t_0) = e_i \) (\( i \)-th column of \( I_n \)). The trajectory does not leave the orthant \( \mathbb{R}^n_+ \) only if \( x(t) = A(t)e_i \in \mathbb{R}^n_+ \) what implies \( a_{ij} \geq 0 \) for \( i \neq j \) and \( A(t) \in M_n \), \( t \in [0, +\infty) \). From the same reason for \( x(t_0) = 0 \) we have \( x(t) = B(t)u(t) \in \mathbb{R}^n_+ \) what implies \( B(t) \in \mathbb{R}^{n \times m}_+ \) for \( t \in [0, +\infty) \) since \( u(t) \in \mathbb{R}^m_+ \) can be arbitrary. From (18b) for \( u(t) = 0 \), \( t \in [0, +\infty) \) we have \( y(t_0) = C(t)x(t_0) \in \mathbb{R}^p_+, \quad t > t_0 \) and \( C(t) \in \mathbb{R}^{p \times n}_+, \quad t \in [0, +\infty) \) since \( x(t_0) \in \mathbb{R}^n_+ \) can be arbitrary. In a similar way assuming \( x(t_0) = 0 \) we obtain \( y(t) = D(t)u(t) \in \mathbb{R}^p_+, \quad t > t_0 \) and \( D(t) \in \mathbb{R}^{p \times m}_+ \) since \( u(t) \in \mathbb{R}^m_+ \) is arbitrary. □

**Example 1.** Consider the positive time-varying continuous-time linear system (1a) with the matrices
\[
A(t) = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -e^{-t} & 0 \\ 1 & e^{-t} & -e^{-t} \end{bmatrix} \quad B(t) = \begin{bmatrix} 1.1 + \sin t \\ -e^{-t} \\ 1.2 + \cos t \end{bmatrix}
\] (24)
The system described by (18a) with the matrices (24) is positive since \( A(t) \in M_3 \) and \( B(t) \in \mathbb{R}^3_+ \) for \( t \in [0, +\infty) \).
4. ASYMPTOTIC STABILITY OF POSITIVE TIME-VARYING LINEAR SYSTEMS

Consider the positive time-varying autonomous linear system

$$\dot{x}(t) = A(t) x(t)$$

(25)

where $x(t) \in \mathbb{R}_+^n$ and $A(t) = [a_{ij}(t)]_{i,j=1,...,n} \in M_n$.  

**Definition 2.** The positive time-varying system (25) is called asymptotically stable if

$$\lim_{t \to \infty} x(t) = 0 \text{ for all } x(0) \in \mathbb{R}_+^n.$$  

(26)

**Theorem 3.** The positive time-varying system (25) is asymptotically stable if there exists a strictly positive vector $c = [c_1 \ldots c_n]^T$ with $c_k > 0$, $k = 1,2,\ldots,n$ such that

$$A(t)c < 0 \text{ for all } t \in [0,\infty).$$

(27)

**Proof.** Sufficiency will be accomplished by the use of Lyapunov method. As a candidate for Lyapunov function $V(x(t))$ we choose

$$V(x(t)) = c^T x(t)$$

(28)

which is positive for $x(t) \in \mathbb{R}_+^n$, $t \in [0,\infty)$. Note that the positive system (25) is asymptotically stable if and only if the positive system

$$\dot{x}(t) = A^T(t)x(t)$$

(29)

is asymptotically stable.

From (28) and (29) we obtain

$$V(x(t)) = c^T \dot{x}(t) = c^T A^T(t)x(t) = x^T(t) A(t)c < 0 \text{ for } t \in [0,\infty)$$

if the condition (27) is satisfied.

Necessity will be shown by contradiction. Assume that the system (25) is asymptotically stable but the condition (27) is not satisfied. Let for some $t_1 \in [0,\infty)$, $c = x(t_1) \in \mathbb{R}_+^n$. Then form (25) we have

$$\dot{x}(t_1) = A(t_1)c < 0$$

(31)

since by assumption the system is asymptotically stable and $-x(t_1) \in \mathbb{R}_+^n$. Therefore, we obtain the contradiction and the system (25) is asymptotically stable only if the condition (27) is satisfied. □

**Theorem 4.** The positive system (25) is asymptotically stable if and only if one of the following conditions is satisfied:

All coefficients of the polynomial

$$\det[I_n s - A(t)] = s^n + a_{n-1}(t)s^{n-1} + \ldots + a_1(t)s + a_0(t),$$

(32)

are positive, i.e. $a_k(t) > 0$, $k = 0,1,\ldots,n-1$ and $t \in [0,\infty)$. All leading principle minors $M_k(t)$, $k = 1,\ldots,n$ of the matrix $-A(t)$ are positive, i.e.

$$M_1(t) = -a_{n-1}(t) > 0, \quad M_2(t) = \begin{vmatrix} -a_1(t) & -a_2(t) \\ -a_2(t) & -a_3(t) \end{vmatrix} > 0,$$

$$\ldots, M_n(t) = \det[-A(t)] > 0$$

The equivalence of the conditions and the condition (27) can be shown in a similar way as for time-invariant linear systems [17].

**Theorem 5.** If the positive system (25) is asymptotically stable then its matrix $A(t)$ satisfies the condition

$$-A^{-1}(t) \in \mathbb{R}^{n \times n}_+ \text{ for } t \in [0,\infty).$$

(34)

Proof will be accomplished by induction. By Theorem 4 the positive system (25) is asymptotically stable if and only if the conditions (33) are satisfied.

From (33) for $n = 1$ we have $A_1^{-1}(t) = -\frac{1}{a_{1}(t)} > 0$. Similarly from (33) for $n = 2$ we obtain

$$-A_2^{-1}(t) = \begin{bmatrix} -a_1(t) & -a_2(t) \\ -a_2(t) & -a_3(t) \end{bmatrix}^{-1} = \frac{1}{a_1(t)a_3(t) - a_2(t)^2} \begin{bmatrix} a_2(t) & a_3(t) \\ a_3(t) & a_1(t) \end{bmatrix} \in \mathbb{R}^{2 \times 2}_+$$

(35)

since $a_1(t)a_3(t) - a_2(t)^2 > 0$ and $a_{11}(t) > 0$, $a_{22}(t) > 0$, $a_{12}(t) > 0$ for $t \in [0,\infty)$. Assuming that $-A_{n-1}^{-1}(t) \in \mathbb{R}^{(n-1) \times (n-1)}_+$, $t \in [0,\infty)$ we shall show that $-A_n^{-1}(t) \in \mathbb{R}^{n \times n}_+$, $t \in [0,\infty)$. It is easy to check that the inverse matrix of the matrix

$$-A_n(t) = \begin{bmatrix} -a_{n-1}(t) & -a_{n}(t) \\ -v_n(t) & -a_n(t) \end{bmatrix} \quad u_n(t) = \begin{bmatrix} a_{n-1}(t) \\ \vdots \\ a_{n-1}(t) \end{bmatrix},$$

(36)

has the form

$$-A_n^{-1}(t) = \begin{bmatrix} -a_{n-1}(t) + \frac{A^{-1}_{n-1}(t)u_n(t)v_n(t)A_{n-1}(t)}{a_n(t)} & \frac{A^{-1}_{n-1}(t)u_n(t)}{a_n(t)} \\ \frac{v_n(t)A^{-1}_{n-1}(t)}{a_n(t)} - \frac{1}{a_n(t)} \end{bmatrix}$$

(37a)

where

$$a_n(t) = a_{n-1}(t) - v_n(t)A_{n-1}^{-1}(t)u_n(t).$$

(37b)

From assumption $-A_{n-1}^{-1}(t) \in \mathbb{R}^{(n-1) \times (n-1)}_+$, $t \in [0,\infty)$ and (37b) it follows that $a_n(t) > 0$ and $-\frac{A^{-1}_{n-1}(t)u_n(t)}{a_n(t)} \in \mathbb{R}^{n-1}_+$.

$$\begin{bmatrix} -v_n(t)A^{-1}_{n-1}(t) \\ a_n(t) \end{bmatrix} \in \mathbb{R}^{n-1}_+,$$

$$A_n(t) = \begin{bmatrix} -2e^t & e^{-t} & 0 \\ e^{-t} & -3e^{-2t} & -e^t \\ 0 & e^{-t} & -2 \end{bmatrix}$$

(38)

The system (25) with (38) is positive since matrix (38) is a Metzler matrix for $t \in [0,\infty)$. By Theorem 3 the system is also asymptotically stable since for $c = [1 \ 1 \ 1]^T$ we obtain

$$A(t)c = \begin{bmatrix} -2e^t & e^{-t} & 0 \\ e^{-t} & -3e^{-2t} & -e^t \\ 0 & e^{-t} & -2 \end{bmatrix} = \begin{bmatrix} -2e^t + e^{-t} \\ -3e^{-2t} + 2e^t \\ -2 + e^{2t} \end{bmatrix} < 0 \quad (39)$$

for $t \in [0,\infty)$. The same result follow from Theorem 4, since all coefficients of the polynomial

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are positive for $t \in [0, +\infty)$ and the leading principle minors of the matrix $-A(t)$

$$
M_1(t) = 2e^t > 0, \quad M_2(t) = \begin{bmatrix}
2e^t & -e^{-t} \\
-e^{-t} & 3e^{-2t}
\end{bmatrix} = 6e^{-t} - e^{-2t} > 0,
$$

(40)

$$
M_3(t) = \det[-A(t)] = 10e^{-t} - 2e^{-2t} > 0
$$

for $t \in [0, +\infty)$.

The inverse matrix $-A^{-1}(t)$ of the matrix (38) has the form

$$
-A^{-1}(t) = \frac{1}{10e^{-t} - 2e^{-2t}} \begin{bmatrix}
5e^{-2t} & 2e^t & e^{-2t} \\
2e^{-t} & 4e^t & 2 \\
e^{-2t} & 2 & 6e^{-t} - e^{-2t}
\end{bmatrix}
$$

(42)

and all its entries are positive for $t \in [0, +\infty)$.

5. POSITIVE TIME-VARYING LINEAR ELECTRICAL CIRCUITS

**Example 3.** Consider the time-varying electrical circuit shown in Fig. 1 with given nonzero resistances $R_1(t)$, $R_2(t)$, $R_3(t)$ inductances $L_1(t)$, $L_2(t)$, $L_3(t)$ capacitances $C_1(t)$, $C_2(t)$, $C_3(t)$ depending on time $t$, and source voltages $e_1(t)$, $e_2(t)$.

\[ \text{Fig. 1. Electrical circuit} \]

Using Kirchhoff’s laws, we can write the equation

$$
e_1(t) = \begin{bmatrix}
R_1(t) + \frac{dL_1(t)}{dt} \\
R_2(t) + \frac{dL_2(t)}{dt} \\
R_3(t) + \frac{dL_3(t)}{dt}
\end{bmatrix} \begin{bmatrix}
i_1(t) \\
i_2(t) \\
i_3(t)
\end{bmatrix} + \begin{bmatrix}
\frac{L_1(t)}{2} \\
\frac{L_2(t)}{2} \\
\frac{L_3(t)}{2}
\end{bmatrix} + \begin{bmatrix}
e_1(t) \\
e_2(t) \\
e_3(t)
\end{bmatrix},
$$

(43)

which can be written in the form

$$
\frac{d}{dt} \begin{bmatrix}
i_1(t) \\
i_2(t) \\
i_3(t)
\end{bmatrix} = A(t) \begin{bmatrix}
i_1(t) \\
i_2(t) \\
i_3(t)
\end{bmatrix} + B(t) \begin{bmatrix}
e_1(t) \\
e_2(t) \\
e_3(t)
\end{bmatrix}
$$

(44a)

where

$$
A(t) = \begin{bmatrix}
\frac{R_1(t) + R_2(t) + \frac{dL_1(t)}{dt}}{L_1(t)} & \frac{R_1(t)}{L_1(t)} & \frac{R_1(t) + \frac{dL_1(t)}{dt}}{L_1(t)} \\
0 & \frac{R_2(t) + \frac{dL_2(t)}{dt}}{L_2(t)} & \frac{R_2(t) + \frac{dL_2(t)}{dt}}{L_2(t)} \\
\frac{1}{L_3(t)} & \frac{R_3(t) + \frac{dL_3(t)}{dt}}{L_3(t)} & \frac{R_3(t) + \frac{dL_3(t)}{dt}}{L_3(t)}
\end{bmatrix}
$$

(44b)

$$
B(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1}{L_3(t)}
\end{bmatrix}
$$

From (44b) it follows that for $R_1(t) > 0$, $R_2(t) > 0$, $R_3(t) > 0$, $L_1(t) > 0$, $L_2(t) > 0$, and $e_1(t) \geq 0$, $e_2(t) \geq 0$ for $t \in [0, +\infty)$ the matrix $A(t) \in M_2$ and $B(t) \in \mathbb{R}^{2 \times 2}$ for $t \in [0, +\infty)$. Therefore, the electrical circuit is a positive time-varying system.

The positive electrical circuit is also asymptotically stable. Using the condition (27) for $c = [1 \ 1]^T$ and (44b) we obtain

$$
A(t)c = \begin{bmatrix}
- \frac{R_1(t) + \frac{dL_1(t)}{dt}}{L_1(t)} & \frac{R_1(t)}{L_1(t)} & \frac{R_1(t) + \frac{dL_1(t)}{dt}}{L_1(t)} \\
0 & - \frac{R_2(t) + \frac{dL_2(t)}{dt}}{L_2(t)} & \frac{R_2(t) + \frac{dL_2(t)}{dt}}{L_2(t)}
\end{bmatrix} \begin{bmatrix}1 \\
1
\end{bmatrix}
$$

(45)

Therefore, by Theorem 3 the positive electrical circuit is asymptotically stable if $R_1(t) + \frac{dL_1(t)}{dt} > 0$ for $k = 1, 2$. The same result we obtain by the use of the conditions of Theorem 4.

**Example 4.** Consider the time-varying electrical circuit shown in Fig. 2 with given nonzero resistances $R_1(t)$, $R_2(t)$, $R_3(t)$, inductance $L(t) > 0$, capacitance $C(t) > 0$ and source voltage $e(t) \geq 0$ for $t \in [0, +\infty)$.

\[ \text{Fig. 2. Electrical circuit} \]

It will be shown that the electrical circuit is a positive time-varying linear system if and only if $R_1(t) = 0$ for $t \in [0, +\infty)$.

Using Kirchhoff’s laws, we can write the equation

$$
e(t) = R_1(t) \left[ i(t) + C(t) \frac{du(t)}{dt} + \frac{dC(t)}{dt} u(t) \right]
$$

$$
+ R_2(t) \left[ C(t) \frac{du(t)}{dt} + \frac{dC(t)}{dt} u(t) \right] + u(t),
$$

(46)

$$
e(t) = R_1(t) \left[ i(t) + C(t) \frac{du(t)}{dt} + \frac{dC(t)}{dt} u(t) \right]
$$

$$
+ \left[ L(t) \frac{di(t)}{dt} + \frac{dL(t)}{dt} i(t) + R_3(t)i(t),
$$

(47a)

which can be written in the form

$$
\frac{d}{dt} \begin{bmatrix}
i(t) \\
u(t)
\end{bmatrix} = A(t) \begin{bmatrix}
i(t) \\
u(t)
\end{bmatrix} + B(t)e(t)
$$

(47a)

where
\[
\mathbf{A}(t) = \begin{bmatrix} 0 & [R(t) + R(t)]C(t) \end{bmatrix}^{-1}
\]

\[
\times \begin{bmatrix} -R(t) & -[R(t) + R(t)]dC(t) \frac{dt}{dt} - 1 \\
-R(t) - R(t) & -R(t) \frac{dt}{dt} 
\end{bmatrix}
\]

\[
\begin{bmatrix} R^2(t) & \frac{R(t)}{[R(t) + R(t)]L(t)} \\
- \frac{R(t)}{[R(t) + R(t)]C(t)} & \frac{R(t)}{[R(t) + R(t)]C(t)} + 1
\end{bmatrix}
\]

\[
\mathbf{B}(t) = \begin{bmatrix} 0 & [R(t) + R(t)]C(t) \end{bmatrix}^{-1}
\]

\[
\frac{R(t)}{[R(t) + R(t)]L(t)}
\]

From (47b) it follows that \( \mathbf{A}(t) \in M \) if and only if \( R(t) = 0 \) for \( t \in [0, +\infty) \). Therefore, the electrical circuit is a positive time-varying system if and only if \( R(t) = 0 \) for \( t \in [0, +\infty) \). Note that the matrix (47b) is diagonal with negative entries if and only if \( R(t) = 0 \), \( t \in [0, +\infty) \) and the positive electrical circuit is asymptotically stable for \( R(t) + \frac{dL(t)}{dt} > 0 \) and \( R(t) + \frac{dC(t)}{dt} > 0 \), \( t \in [0, +\infty) \).

Now let us consider electrical circuit shown on Fig. 3 with given positive resistances \( R_k(t) \), \( k = 0, 1, ..., n \), inductances \( L_i(t) \), \( i = 2, 4, ..., n_2 \), capacitances \( C_j(t) \), \( j = 1, 3, ..., n_1 \) depending on time \( t \) and source voltages \( e_1(t), e_2(t), ..., e_n(t) \). We shall show that this electrical circuit is a positive and asymptotically stable time-varying linear system.

Using Kirchhoff's law we can write the equations

\[
e_1(t) = R_1(t)C_1(t) \frac{du_1(t)}{dt} + R_1(t) \frac{dC_1(t)}{dt} + [u_1(t)] \quad (48a)
\]

for \( k = 1, 3, ..., n_1 \),

\[
e_1(t) + e_2(t) = L_2(t) \frac{di_2(t)}{dt} + R_2(t) + \frac{dL_2(t)}{dt} \quad i_2(t) + u_2(t) \quad (48b)
\]

for \( k = 2, 4, ..., n_2 \),

which can be written in the form

\[
\frac{d[u(t)]}{dt} = \mathbf{A}(t) \begin{bmatrix} u(t) \\ i(t) \end{bmatrix} + B(t)e(t) \quad , (49a)
\]

where

\[
u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ ... \\
... \\
u_n(t) \end{bmatrix}, \\
i(t) = \begin{bmatrix} i_1(t) \\ i_2(t) \\ ... \\
... \\
i_n(t) \end{bmatrix}, \\
e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ ... \\
... \\
e_n(t) \end{bmatrix}, \quad (n = n_1 + n_2) (49b)
\]

and

\[A(t) = \text{diag}[-a_1(t), -a_2(t), ..., -a_n(t)], \quad a_k(t) = \frac{R_k(t)\frac{dC_k(t)}{dt} + 1}{R_k(t)C_k(t)} \quad \text{for} \quad k = 1, 3, ..., n_1,
\]

\[a_k(t) = \frac{R_k(t)\frac{dL_k(t)}{dt}}{L_k(t)} \quad \text{for} \quad k = 2, 4, ..., n_2,
\]

\[
B(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}, \\
B_1(t) = \begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \\ 0 \end{bmatrix}, \\
B_2(t) = \begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \\ 0 \end{bmatrix}
\]

The electrical circuit is positive and asymptotically stable time-varying linear system since all diagonal entries of the matrix \( A(t) \) are negative functions of \( t \in [0, +\infty) \) and the matrix \( B(t) \) has nonnegative entries for \( t \in [0, +\infty) \) if \( \frac{dL_k(t)}{dt} \geq 0 \) and \( \frac{dC_k(t)}{dt} \geq 0 \). The solution of the equation (48a) can be found using Lemma 1.

![Fig. 3. Positive and stable electrical circuit.](image-url)
CONCLUDING REMARKS

The positivity and asymptotic stability of time-varying continuous-time linear systems and electrical circuits have been addressed. Necessary and sufficient conditions for the positivity and asymptotic stability of the systems and electrical circuits have been established. It has been shown that there exists a large class of positive and asymptotic stable electrical circuits with time-varying parameters. The considerations have been illustrated by positive and asymptotic stable electrical circuits. The consideration can be extended to fractional time-varying electrical circuits.

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REFERENCES


Dodatnie i stabilne asymptotycznie ciągłe układy i obwody elektryczne o parametrach zmiennych w czasie.

W pracy są analizowane zagadnienia dodatności i stabilności asymptotycznej ciągłych układów i obwodów elektrycznych o parametrach zmiennych w czasie. Podano warunki konieczne i wystarczające dodatności i stabilności asymptotycznej tej klasy układów i obwodów elektrycznych. Zostanie wykazane, że istnieje obszerna klasa dodatnich i stabilnych asymptotycznie obwodów elektrycznych o zmiennych parametrach. Rozważania ogólne zostaną zilustrowane przykładami obwodów elektrycznych o zmiennych w czasie parametrach.

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